Final exam
Spring term 2023

Assignment date:
Due date:

PHYS-314 – Mock Exam

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section.

  Please mark clearly which two you have answered below and **start a new sheet for each of the long answer questions**.
- Write your solutions in the indicated space. Scrap paper will not be corrected.
- **You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.**
- A simple calculator (without internet access) is allowed.
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

NAME STICKER GOES HERE

Short answers: Problem 1	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
Total	/100

Short questions

1. **Quantum States.**

   a) State the three properties an operator ρ must satisfy in order to be a density operator. Deduce from these properties the conditions under which the operator

   $$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

   is a density operator, where the set of states $\{|\psi_n\rangle\}$ are orthonormal.

   (3 marks)

   b) Explain in terms of a density operator ρ the meaning of the terms *pure state* and *mixed state*. Show that $\mathrm{Tr}(\rho^2) = 1$ if and only if ρ is pure.

   (3 marks)

   c) A composite system consisting of components A, B has Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and its state is

   $$|\psi\rangle = \sum_n \alpha_n |a_n\rangle \otimes |b_n\rangle,$$

   where $\{|a_n\rangle\}$ and $\{|b_n\rangle\}$ are orthonormal sets of states. Compute ρ_A, the reduced density operator of system A obtained by tracing over B. Compute ρ_B, defined similarly. Hence show that:

   $$\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2)$$

   (4 marks)

2. **Fermions and Bosons.**

   a) What properties does a state describing N indistinguishable particles $|\psi\rangle$ need to fulfil if the particles are Bosons? How does this change if the particles are Fermions?

   (3 marks)

   b) Consider a three particle state where each particle can be in one of the three states $|a\rangle$, $|b\rangle$, $|c\rangle$. Write down the allowed states of the system if these particles are i. Bosons and ii. Fermions.

   (6 marks)

3. **Symmetry**

   a) Write down the Cayley table for the permutation group S_3?

   (4 marks)

   b) Describe a physical systems that obeys this symmetry. Write down a representation for the S_3 group that implements the symmetry transformations of this system.

(3 marks)

c) What are the conjugacy classes of S_3? Write down the corresponding character table.

(4 marks)

4. **Perturbation Theory.**

Consider a Hamiltonian of the form :

$$\hat{H} = \hat{H}_0 + \lambda \hat{V} \tag{1}$$

with :

$$\hat{H}_0 = \epsilon_1 |1\rangle \langle 1| + \epsilon_2 |2\rangle \langle 2| \quad \text{et} \quad \hat{V} = V_{12} |1\rangle \langle 2| + V_{21} |2\rangle \langle 1| \tag{2}$$

where $|1\rangle$ and $|2\rangle$ are the eigenstates of $\hat{H}_0$ associated respectively with the eigenvalues ϵ_1 and ϵ_2. We assume that $\Delta \equiv \epsilon_2 - \epsilon_1 > 0$ and that V_{12} and V_{21} are real.

a) Calculate the exact spectrum of the Hamiltonian as well as its eigenstates.

(5 marks)

b) Using first-order perturbation theory, find the eigenstates and compare with the exact result.

(6 marks)

5. **Decoherence.**

Consider a composite system that is prepared in the initial state $|\psi\rangle = \sum_j c_j |E_j\rangle_A \otimes |\phi\rangle_B$ and evolves under a Hamiltonian $H_{AB} = \sum_j |E_j\rangle\langle E_j|_A \otimes H_B^{(j)}$ for time t.

a) Find an expression for the reduced states $\rho_A(t)$ and $\rho_B(t)$ of systems A and B as a function of time.

(4 marks)

b) Under what circumstances do A and B remain pure for all times?

(3 marks)

c) Under what circumstances does $\rho_A(t)$ become approximately diagonal in the basis $\{|E_j\rangle\}$?

(2 marks)

Long questions

Please **pick 2 questions** to attempt - mark your choices clearly on the cover sheet.

Start a new sheet for each question.

Question A- Variational Principle

Consider the problem of a particle in one dimension, defined by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \hat{V}(x) \tag{3}$$

The potential $\hat{V}(x)$ takes the form of a well, i.e., $\hat{V}(x) \leq 0 \;\; \forall x$, and $\hat{V}(x) \to 0$ as $|x| \to \infty$. Use the variational principle and the wavefunction ansatz $\psi(x) = A\exp(-\lambda x^2)$, which depends on the variational parameter $\lambda > 0$, to show that there is always at least one bound eigenstate, i.e., with eigenenergy $E_0 < 0$. In particular,

1. State the variational principle and explain how it can be used to estimate the ground state energy of a Hamiltonian H.

   (3 marks)

2. Calculate the normalization factor A.

   (4 marks)

3. Calculate $\langle\psi|\hat{T}(x)|\psi\rangle = \langle\psi|\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)|\psi\rangle$.

   (6 marks)

4. We denote $I(\lambda) = \langle\psi|\hat{V}(x)|\psi\rangle$. So $\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\hat{T}(x)|\psi\rangle + I(\lambda)$. Explicitly write the condition that minimizes the expectation of energy $\langle\psi|\hat{H}|\psi\rangle$. Use the resulting relation to derive an expression for $I(\lambda)$. Use this result in the expression for $\langle\psi|\hat{H}|\psi\rangle$ and demonstrate that there always exists a $|\psi\rangle$ such that $\langle\psi|\hat{H}|\psi\rangle < 0$.

   (9 marks)

5. Explain how the variational principle can be used to find an estimate of the energy of the first excited state of a Hamiltonian. State any limitations of this approach.

   (3 marks)

You may find the following integrals helpful:

$$\int_{-\infty}^{+\infty} dx \exp(-x^2) = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

Question B- Symmetry

1. Prove that the Pauli matrices and the identity (times $\pm 1,\ \pm i$) form a (non-Abelian) group with the matrix product.

   (2 marks)

2. Prove that if $R(g)$ is a representation of a group G then $R(g) \otimes R(g)$ is also a representation of G.

   (2 marks)

3. Consider a unitary irreducible representation $R(g) = U_g$ of group G. Use the Grand Orthogonality Theorem to prove that

$$\frac{1}{N} \sum_g U_g X U_g^\dagger = \frac{1}{d} \operatorname{Tr}[X]\, I \tag{4}$$

   where $d = \dim(X)$ and N is the order of the group.

   (4 marks)

4. Use this result to (carefully!) explain why randomly applying either I (i.e, do nothing), σ_x, σ_y, or σ_z (with equal probability) to any single qubit state on average results in the maximally mixed state.

   (3 marks)

5. Consider now instead a completely reducible unitary representation $U_g = \oplus_k R_k(g)$ where the $R_k(g)$ are d_k dimensional unitary irreducible representations. It can be shown that

$$\langle X \rangle_G = \frac{1}{N} \sum_g U_g X U_g^\dagger = \frac{1}{d_k} \bigoplus_k \operatorname{Tr}[X \Pi_k] \Pi_k \,. \tag{5}$$

   What are Π_k and d_k in this expression?

   (3 marks)

6. The above relation for averaging over representations of finite groups, Eq. (5), generalizes to averaging over compact Lie groups. In this case the finite average $\frac{1}{N} \sum_g$ becomes a continuous integral over a uniform measure $\int d\mu(g)$ and we have:

$$\langle X \rangle_G := \int_G d\mu(g) U_g X U_g^\dagger = \frac{1}{d_k} \bigoplus_k \operatorname{Tr}[X \Pi_k] \Pi_k. \tag{6}$$

   Use this result to derive an explicit expression (i.e. compute the relevant d_k and Π_k) for the averaged state ρ that results from randomly evolving ρ under the tensor product of two random single qubit unitaries. That is, from apply $U \otimes U$ with $U \in \mathrm{U}(2)$, to any two qubit state ρ, and then averaging:

$$\langle \rho \rangle = \int_{U(2)} d\mu \, U \otimes U \, \rho \, U^\dagger \otimes U^\dagger \; . \tag{7}$$

(5 marks)

7. Hence (or otherwise) compute the states that result from averaging (i.e, compute $\langle \rho \rangle$ in Eq. (7)) for the following states:

   i. $\rho = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

   ii. $\rho = |\Phi^-\rangle\langle\Phi^-|$ with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

   iii. $\rho = |00\rangle\langle 00|$

   iv. An arbitrary tensor product two qubit state $\rho \otimes \sigma$ (hint: use the Bloch vector representation).

   (6 marks)

Question C - Entanglement

A quantum system is made of 2 sub-systems and is defined in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the spaces where the 2 sub-systems ares defined. The states of such a system is said to be separable if we can write its density matrix as

$$\rho_s = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}, \tag{8}$$

with $\sum_k p_k = 1$, $p_k \geq 0$, and $\rho_k^{(1)}$ and $\rho_k^{(2)}$ being density matrices in spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. A system that cannot be described by a matrix of the form of Eq. (8) is a system with quantum entanglement. (Recall that a density matrix must satisfy the following properties: (i) $\mathrm{Tr}(\rho) = 1$; (ii) $\rho = \rho^\dagger$; (iii) is positive semi-definite.)

1. Show that, for such a separable state, the mean value of an arbitrary observable quantity A_1 of subsystem 1, does not depend on subsystem 2. That is, does not depend on the $\rho_k^{(2)}$.

   (4 marks)

Three players (called Alice, Bob and Charlie) each own a qubit (a quantum system defined in a Hilbert space of dimension 2, with basis $\{|0\rangle, |1\rangle\}$). The three qubit system is in state $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ (For the rest of the problem, the notation $|ijk\ldots\rangle$, i indicates the qubit state of Alice, j that of Bob, etc.). Alice lives in another galaxy, Bob and Charlie have no knowledge of the total system state.

2. Calculate the mixed state density matrix that describes the subsystem formed by the qubits of Bob and Charlie (In basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$).

   (2 marks)

3. Show that this state is separable. (1 marks)

We now consider the partial transpose operation (not the partial trace). Consider a density matrix ρ describing the state of a system made up of two sub-systems. We indicate with $\{|i\rangle, |j\rangle, \ldots\}$ the states of the basis of the first sub-system; with $\{|\mu\rangle, |\nu\rangle, \ldots\}$ the basis of the second sub-system; and with $\{|i\mu\rangle, |i\nu\rangle, |j\mu\rangle, |j\nu\rangle, \ldots\}$ the total system. If the matrix ρ has elements $\rho_{i\mu,j\nu} = \langle i\mu|\rho|j\nu\rangle$, then the elements of the density matrix ρ^{T_P}, obtained by partial transpose wrt to the subsystem, are defined by $(\rho^{T_P})_{i\mu,j\nu} = \langle i\nu|\rho|j\mu\rangle$.

4. Show that for a separable state of two subsystems, of the form (8), the partial transpose $\rho_s^{T_P}$ with respect to one of the two subsystems is still a valid density matrix. In other words, it still satisfies the three properties (i), (ii), and (iii) mentioned above.

   (4 marks)

5. Hence explain how the partial transpose can be used to determine whether or not a mixed state is entangled.

   (4 marks)

Four actors, named A, B, C, and D (or Alice, Bob, Charlie, and David), each have a quantum bit. The system of the four quantum bits is in the state $|\psi_S\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$. As before, Alice lives in another galaxy. Calculate the density matrix associated with the mixed state that describes the subsystem formed by the quantum bits of Bob, Charlie, and David (in the basis $\{|000\rangle, |001\rangle, |010\rangle, \ldots, |111\rangle\}$).

6. Demonstrate that the mixed state shared by Bob, Charlie, and David is an entangled state. (10 marks)